ETMAG LECTURE 4

Limits of sequences

Subsequences.

Definition. If $(a_n)_{n=1}^{\infty}$ is a sequence, then for every increasing sequence of natural numbers $(n_k)_{k=1}^{\infty}$ the sequence $(b_k)_{k=1}^{\infty} = (a_{n_k})_{k=1}^{\infty}$ is called a *subsequence* of $(a_n)_{n=1}^{\infty}$. Usually, we simply write "(a_{n_k}) is a subsequence of (a_n)".

Thus, a subsequence of (a_n) is a sequence obtained by the removal from (a_n) some (possibly none, possibly infinitely many, but leaving infinitely many) of its terms without changing the order of the remaining terms.

Examples.

1. Every sequence is its own subsequence.
2. (1,2,3,4) is not a subsequence of (n) because it is finite.
3. (3,1,5,11,7,9, …etc.) is not a subsequence of (n). Odd positive integers do form an infinite subset of $\mathbb{N}$, but the order of terms is messed-up.

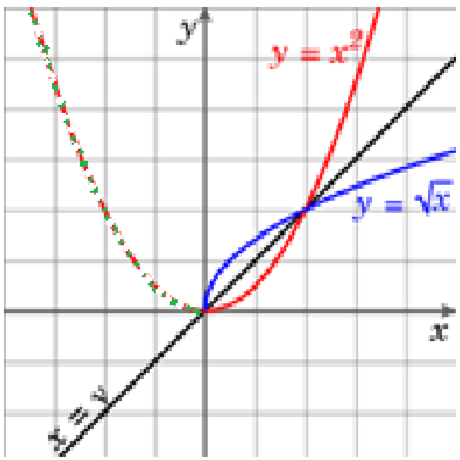
4. (1,3,5,7,9, … etc.) is a subsequence of (n) defined by $n_k = 2k + 1$ – an increasing sequence of odd natural numbers.
5. (p_n) = (2,3,5,7,11,13, … (*all primes*)) is a subsequence of (n) we cannot provide the explicit formula for n_k (other than n_k is the *k*-th prime).
6. (a_{p_n}) is the subsequence of (a_n) of all terms whose indices are primes. This yields $(a_2, a_3, a_5, a_7, a_{11}, \ldots\ etc)$.
7. The sequence(a_{2n}) is the subsequence of (a_n) consisting of all even-subscripted terms.

Definition.

Let $f: X \to Y$ be a function and let A be a subset A of X . The *restriction* of f to A is the function $f|_A: A \to Y$ whose domain is A and for each $a \in A$ $f|_A(a) = f(a)$.

Example.

$x^2|_{[0;\infty)}$ becomes a 1-1 function.



The solid red line is the graph of the function f(x)=x^2 restricted to <0;∞). The restricted function is 1-1 (the original is not), hence invertible. The blue line is the graph of its inverse, $\sqrt{x}$.

The spotted red line is the graph of x^2 restricted to $(-\infty; 0>$. (*Picture from Wikipedia*).

Since a sequence is a function from $\mathbb{N}$ into $\mathbb{R}$ you may think of a subsequence of a sequence as a restriction of the sequence to an infinite subset of $\mathbb{N}$. („*As a what of what to what*?!" – Winnie the Pooh).

Comprehension.

What type of sequence is

(a) (a_{2n}) if (a_n) is an arithmetic sequence with the increment d,

(b) (a_{2n}) if (a_n) is a geometric sequence with the quotient q,

(c) $(a_{2^n})_{n=0}^{\infty}$ if (a_n) is the arithmetic sequence with the increment d and $a_1=d$?

Limits of sequences

Definition.
Let (a_n) be a sequence. We say that the sequence is *convergent* if and only if there exists a number L such that

$$(\forall \varepsilon > 0)(\exists p \in \mathbb{N})(\forall n > p)|a_n - L| < \varepsilon.$$

We call L the limit of (a_n), in symbols $L = \lim_{n\to\infty} a_n$, and we say that a_n *converges to* L.

If no such number exists, the sequence is called *divergent*.

The standard student reaction to the definition is "does it have to be this complicated". The standard teacher's answer is YES. We cannot say "the greater n the closer we are to L" because there may be "local fluctuations" on our approach to L. Like when you drive somewhere there might be detours, bridges to cross, MacDonalds' to visit and other obstacles and temptations which cause your distance from the destination (measured as the crow flies) to temporarily increase. But eventually you get there. Hopefully.

Which is to say that $d_n = |a_n - L|$ decreases to zero is not good enough.

Notice that the sequence is convergent to L if, whenever somebody chooses an $\varepsilon>0$, you can find such an index *p* that all terms of the sequence with indices greater than *p* belong to the interval $(L-\varepsilon; L+\varepsilon)$. In other words, only finitely many terms of (a_n) may sit outside this interval – namely $a_1, a_2, \ldots$ and a_p at the most.

Example 1.

Consider the sequence $a_n = \frac{1}{n}$. Intuitively, $\frac{1}{n}$ approaches 0 as n approaches ∞.

Formally, choose (but do not disclose) $\varepsilon > 0$. Can we find $p \in \mathbb{N}$ such that $n > p$ guarantees $|a_n - 0| = |1/n| = \frac{1}{n} < \varepsilon$, regardless of the actual value of ε? Certainly, this hypothetical p must depend on ε.

$\frac{1}{n} < \varepsilon$ is equivalent to $\frac{1}{\varepsilon} < n$. Hence, if we put p to be any natural number greater than $\frac{1}{\varepsilon}$, for example $\left\lceil\frac{1}{\varepsilon}\right\rceil$ then for every $n > p$ we obtain $\frac{1}{n} < \varepsilon$, as required. Conclusion:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

In this example, the distance between a_n and the limit L decreases steadily to 0.

Example 2.

Consider the sequence $a_n = \frac{1}{n} + \frac{2(-1)^n}{n}$. The initial terms of the sequence look like this: $-1, \frac{3}{2}, -\frac{1}{3}, \frac{3}{4}, -\frac{1}{5}, \frac{3}{6}, -\frac{1}{7}, \frac{3}{8},$ etc. As you see, the distance to 0 (which is suspected of being the(?) limit) do not decrease to 0.

Again, choose (and keep secret) some $\varepsilon > 0$. Can we find $p \in \mathbb{N}$ such that $n > p$ guarantees $|a_n - 0| =< \varepsilon$, regardless of the actual value of ε? $|a_n - 0| = |\frac{1}{n} + \frac{2(-1)^n}{n}| \leq \left|\frac{1}{n}\right| + \left|\frac{2(-1)^n}{n}\right| = \frac{1}{n} + \frac{2}{n} = \frac{3}{n}$. As before, $\frac{3}{n} < \varepsilon$ is equivalent to $\frac{3}{\varepsilon} < n$. Hence, if we put p to be any natural number greater than $\frac{3}{\varepsilon}$, for example $\left\lceil\frac{3}{\varepsilon}\right\rceil$ then for every $n > p$ we obtain $\frac{3}{n} < \varepsilon$, as required, which confirms that:

$$\lim_{n\to\infty} \frac{1}{n} + \frac{2(-1)^n}{n} = 0.$$

Example 3.

Consider the sequence $a_n = (-1)^n$. Suppose it is convergent to some number L. Then, for every positive number ε, in particular for $\varepsilon = \frac{1}{4}$, there exists p such that whenever $n > p$, $a_n \in (L - \frac{1}{4}; L + \frac{1}{4})$. So, if $a_n \in (L - \frac{1}{4}; L + \frac{1}{4})$ then $a_{n+1} \in (L - \frac{1}{4}; L + \frac{1}{4})$. But one of a_n and a_{n+1} is 1 and the other is -1. They cannot both belong to any interval of length $\frac{1}{2}$. This proves that the sequence is divergent.

FAQ 1. Is it true that a sequence is convergent to L iff for every ε infinitely many of its terms are "ε - close" to L?

Of course not. Look at example 3 above.

FAQ 2. Can a sequence have two (or more) different limits?

Theorem.
For every sequence (a_n), if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = K$ then $L = K$.

Proof. (By contradiction)
Suppose $L \neq K$ and take $\varepsilon = \frac{1}{2}|L - K|$. Obviously, $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, all but a finite number of a_n-s belong to the interval $(L - \varepsilon; L + \varepsilon)$. Hence, only a finite number of a_n-s can belong to $(K - \varepsilon; K + \varepsilon)$ i.e., a_n is NOT convergent to K. QED

So, the answer to question 2 is NO.

FAQ 3. Can a convergent sequence have two subsequences converging to two different limits?

Theorem.

For every sequence (a_n), $\lim_{n\to\infty} a_n = L$ iff for every subsequence $\left(a_{n_k}\right)$ of (a_n), $\lim_{k\to\infty} a_{n_k} = L$.

Proof. (⇐) Trivial because "for every subsequence" means, in particular, "for the sequence itself".

(⇒) Let $\varepsilon > 0$ and suppose p is such that $|a_n - L| < \varepsilon$ for all $n>p$. Clearly, for every k, $n_k \geq k$ (because (n_k) is an increasing sequence of subscripts). Hence, for every $k>p$ also $n_k>p$ and $\left|a_{n_k} - L\right| < \varepsilon$. QED

This means the answer to question 3 is NO.

Theorem 3. (Arithmetic properties of the limit)

If sequences (a_n) and (b_n) are convergent then:

1. $(a_n + b_n)$ is convergent and $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

2. $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$,

3. $(a_n b_n)$ is convergent and $\lim\limits_{n\to\infty}(a_n\, b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$,

4. for every constant c ∈ ℝ, (ca_n) is convergent and $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n$

5. $(\frac{a_n}{b_n})$ is convergent and $\lim\limits_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ (if $b_n \neq 0$ and $\lim\limits_{n\to\infty} b_n \neq 0$).

In short, arithmetic operations on convergent sequences *preserve* limits, (the limit of the sum is the sum of limits etc.).

Proof. (Part 1). The starting point is, as always, the fundamental question, "what the hell must we do". We will apply the definition of the limit to the sequence $a_n b_n$. Denote $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$. We must show that

$$(\forall \varepsilon > 0)(\exists p \epsilon \mathbb{N})(\forall n \epsilon \mathbb{N})(p < n \Rightarrow |(a_n + b_n) - (A + B)| < \varepsilon).$$

Now,

$$|a_n + b_n - (A + B)| = |a_n - A + b_n - B| = \leq |a_n - A| + |b_n - B|$$

Since the sequences converge to A and B, respectively, we know that there exist p_a and p_b such that for $n > p_a$, $|a_n - A| < \frac{\varepsilon}{2}$ and for $n > p_b$, $|b_n - B| < \frac{\varepsilon}{2}$. Putting $p = \max(p_a, p_b)$ we obtain that for every $n > p$, $|a_n - A| < \frac{\varepsilon}{2}$ and $|b_n - B| < \frac{\varepsilon}{2}$. Adding the last two inequalities we get $|a_n - A| + |b_n - B| < \varepsilon$. QED

Similar arguments can be used to prove the remaining parts.

Warning. This is an "if…then" NOT "if and only if" theorem!

Theorem 4 (Limits and inequalities)

If a_n is convergent to A and b_n to B and there exists k such that for every $n>k$ $a_n \le b_n$ then A $\le$ B. (*The order is preserved by the limit*).

Outline of a proof by contradiction. Suppose to the contrary, A>B. Put $\varepsilon = \frac{A-B}{2}$. There exists p such that for every $n>p$, $|a_n - A| < \varepsilon$ and $|b_n - B| < \varepsilon$. In other words, for every $n>p$ we have

$-\frac{A-B}{2} < a_n - A < \frac{A-B}{2}$ which implies A $- \frac{A-B}{2} < a_n$, so $\boldsymbol{\frac{A+B}{2} < a_n}$

$-\frac{A-B}{2} < b_n - B < \frac{A-B}{2}$ hence, $b_n - B < \frac{A-B}{2}$ and $b_n < \frac{A+B}{2}$. The red inequalities imply that $b_n < a_n$ for infinitely many n-s, contrary to our assumption. QED,